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El-Nabulsi型非保守动力学系统的近似Noether不变量

王雪萍[1]，张　毅[2]

（1.苏州科技大学数学科学学院，苏州 215009；2.苏州科技大学土木工程学院，苏州 215011）

摘要：为了探究小扰动作用对动力学系统不变量的影响，研究El-Nabulsi指数模型和El-Nabulsi幂律模型下非保守系统的近似Noether不变量。根据Hamilton原理，并以非标准Lagrange函数作为其作用量泛函，建立非保守系统的El-Nabulsi型动力学方程。在此基础上，依据泛函在无限小变换下的不变性，给出非保守系统在小扰动作用下的近似Noether不变量。当未受扰动时，则给出精确Noether不变量。证明了El-Nabulsi指数模型和El-Nabulsi幂律模型下非保守系统的近似Noether不变量定理。本文方法为研究非保守系统动力学提供了一个新的思路，算例亦显示结果之有效性。

关键词：非保守动力学；近似不变量；El-Nabulsi模型；指数Lagrange函数；幂律Lagrange函数

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Approximate Noether Invariants for Nonconservative Dynamical Systems Under El-Nabulsi Models

WANG Xueping[1], *ZHANG Yi*[2]

(1. College of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou 215009, China; 2. College of Civil Engineering, Suzhou University of Science and Technology, Suzhou 215011, China)

Abstract: In order to investigate the influence of small perturbations on the invariants of dynamical systems, we study the approximate Noether invariants of nonconservative systems under El-Nabulsi exponential model and El-Nabulsi power-law model. According to Hamilton's principle with nonstandard Lagrangians as its action functional, the dynamics equations of El-Nabulsi type for nonconservative systems are established. On this basis, due to the invariance of the functional under infinitesimal transformation, the approximate Noether invariants of nonconservative systems under small disturbance are obtained. When not disturbed, the exact Noether invariants are given. The approximate Noether invariant theorems for nonconservative systems under El-Nabulsi exponential model and El-Nabulsi power-law model are proved. This method provides a new idea for the study of nonconservative system dynamics, and two examples show the effectiveness of the results.

Key words: nonconservative dynamics; approximate invariant; El-Nabulsi model; exponential Lagrangians; power-law Lagrangians

自然界和工程中遇到的问题大多是非保守和非线性的。然而，一般而言，非保守系统的Hamilton原理不再是稳定作用量原理。基于经典Hamilton原理，Riewe通过在Lagrange函数中引入分数阶导数项建立了非保守系统的分数阶模型[1]。但是，该模型需要进行复杂的分数阶微积分运算。为此，El-Nabulsi基于Riemann-Liouville分数阶积分的定义提出了非保守Lagrange系统的类分数阶模型[2]，它的优势是方程形似经典非保守动力学方程但不出现分数阶导数项。2013年，El-Nabulsi为处

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通信作者：张毅，男，教授，E-mail：zhy@mail.usts.edu.cn。

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理耗散和非线性问题引入两类非标准 Lagrange 函数[3]。进而，与类分数阶模型相结合，El-Nabulsi 研究了指数 Lagrange 函数的类分数阶模型和幂律 Lagrange 函数的类分数阶模型[4-5]，称之为 El-Nabulsi 指数模型和幂律模型。近年来，关于非保守系统的类分数阶模型和基于非标准 Lagrange 函数的动力学及其对称性的研究已经取得重要进展[6-14]。周小三和张毅[15]研究了 El-Nabulsi 模型下的 Noether 定理。本文进一步研究 El-Nabulsi 模型下非保守动力学系统受小扰动作用的近似 Noether 不变量问题，给出了两类近似不变量。

1　El-Nabulsi 型动力学方程

经典 Hamilton 原理可表为

$$\delta S=0 \tag{1}$$

且满足交换关系

$$\mathrm{d}\delta q_s=\delta \mathrm{d}q_s \tag{2}$$

以及边界条件

$$\delta q_s\big|_{t=t_1}=\delta q_s\big|_{t=t_2}=0 \tag{3}$$

式中：S 为作用量泛函；q_s 为广义坐标，$s=1,2,\cdots,n$。

若作用量[3]取为

$$S_{\mathrm{E}}=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\exp\left[L\left(\tau,q_s,\dot{q}_s\right)\right](t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{4}$$

式中：$L=L\left(\tau,q_s,\dot{q}_s\right)$ 为 Lagrange 函数；Γ 为 Euler 函数；α 为实数或复数；t 为观察者时间，τ 为固有时间，$t\neq\tau$；函数 L 是其变量的 C^2 类函数。由原理(1)，容易导出

$$(t-\tau)^{\alpha-1}\exp(L)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}\tau}+\frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial \dot{q}_s}\right)=0 \tag{5}$$

方程(5)是 El-Nabulsi 指数型动力学方程[13]。

若作用量[4]取为

$$S_{\mathrm{P}}=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left[L^{1+\gamma}\left(\tau,q_s,\dot{q}_s\right)\right](t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{6}$$

则由原理(1)，可导出

$$\left(1+\gamma\right)(t-\tau)^{\alpha-1}L^{\gamma}\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{q}_s}-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}\tau}+\frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial \dot{q}_s}\right)=0 \tag{7}$$

方程(7)是 El-Nabulsi 幂律型动力学方程[13]。

2　精确 Noether 不变量

引进无限小变换

$$\begin{cases}\bar{\tau}=\tau+\varepsilon_\sigma\varsigma^{\sigma0}\left(\tau,q_k,\dot{q}_k\right)\\ \bar{q}_s(\bar{\tau})=q_s(\tau)+\varepsilon_\sigma\xi_s^{\sigma0}\left(\tau,q_k,\dot{q}_k\right)\end{cases}\quad s,k=1,2,\cdots,n \tag{8}$$

式中：$\varepsilon_\sigma(\sigma=1,2,\cdots,r)$ 为无限小参数；$\varsigma^{\sigma0}$、$\xi_s^{\sigma0}$ 为无限小变换的生成元或生成函数。

定理 1[15]　如果生成元 $\varsigma^{\sigma0}$、$\xi_s^{\sigma0}$ 满足 r 个方程

$$\left(\frac{\partial L}{\partial \tau}+\frac{1-\alpha}{t-\tau}\right)\varsigma^{\sigma0}+\frac{\partial L}{\partial q_s}\xi_s^{\sigma0}+\frac{\partial L}{\partial \dot{q}_s}\left(\dot{\xi}_s^{\sigma0}-\dot{q}_s\dot{\varsigma}^{\sigma0}\right)+\dot{\varsigma}^{\sigma0}+\exp(-L)(t-\tau)^{1-\alpha}\dot{G}^{\sigma0}=0\quad \sigma=1,2,\cdots,r \tag{9}$$

式中 $G^{\sigma0}\left(\tau,q_s,\dot{q}_s\right)$ 为规范函数，则

$$I_0^\sigma=\exp(L)\cdot(t-\tau)^{\alpha-1}\left(\varsigma^{\sigma0}+\frac{\partial L}{\partial \dot{q}_s}\left(\xi_s^{\sigma0}-\dot{q}_s\varsigma^{\sigma0}\right)\right)+G^{\sigma0}=\mathrm{const}\quad \sigma=1,2,\cdots,r \tag{10}$$

是 El-Nabulsi 指数型动力学系统(5)的精确 Noether 不变量。

定理 2[15]　如果生成元 $\varsigma^{\sigma0}$、$\xi_s^{\sigma0}$ 满足 r 个方程

$$\left(\frac{\partial L}{\partial \tau}+\frac{L}{1+\gamma}\frac{1-\alpha}{t-\tau}\right)\varsigma^{\sigma0}+\frac{\partial L}{\partial q_s}\xi_s^{\sigma0}+\frac{\partial L}{\partial \dot{q}_s}\left(\dot{\xi}_s^{\sigma0}-\dot{q}_s\dot{\varsigma}^{\sigma0}\right)+\frac{L}{1+\gamma}\dot{\varsigma}^{\sigma0}+\frac{(t-\tau)^{1-\alpha}}{\left(1+\gamma\right)L^{\gamma}}\dot{G}^{\sigma0}=0 \tag{11}$$

则

$$I_0^\sigma=(t-\tau)^{\alpha-1}L^{\gamma}\left(L\varsigma^{\sigma0}+\left(1+\gamma\right)\frac{\partial L}{\partial \dot{q}_s}\left(\xi_s^{\sigma0}-\dot{q}_s\varsigma^{\sigma0}\right)\right)+G^{\sigma0}=\mathrm{const}\quad \sigma=1,2,\cdots,r \tag{12}$$

是 El-Nabulsi 幂律型动力学系统(7)的精确 Noether 不变量。

式(10,12)是未受扰动时动力学系统(5,7)的不变量，它们是由 Noether 对称性导致的。

3　近似 Noether 不变量

如果物理量 $I_z\left(\tau,q_s,\dot{q}_s,\nu\right)$ 中小参数 ν 的最高次幂为 z，而 $\mathrm{d}I_z/\mathrm{d}\tau$ 正比于 ν^{z+1}，则称 I_z 为 z 阶近似不变量。

假设系统受小扰动 νQ_s 的作用，El-Nabulsi 指数型动力学方程(5)成为

$$(t-\tau)^{\alpha-1}\exp(L)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}\tau}+\frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial \dot{q}_s}\right)=\nu Q_s \tag{13}$$

生成元ς^{σ}、ξ_s^{σ}和规范函数G^{σ}相应地成为

$$\begin{aligned}\varsigma^{\sigma}&=\varsigma^{\sigma 0}+\nu\varsigma^{\sigma 1}+\nu^2\varsigma^{\sigma 2}+\cdots\\ \xi_s^{\sigma}&=\xi_s^{\sigma 0}+\nu\xi_s^{\sigma 1}+\nu^2\xi_s^{\sigma 2}+\cdots\\ G^{\sigma}&=G^{\sigma 0}+\nu G^{\sigma 1}+\nu^2 G^{\sigma 2}+\cdots\end{aligned}\tag{14}$$

则有如下定理：

定理3 已知El-Nabulsi指数型动力学系统(5)受小扰动νQ_s的作用，如果生成元ς^{σ_j}、$\xi_s^{\sigma_j}$满足结构方程

$$\left(\frac{\partial L}{\partial\tau}+\frac{1-\alpha}{t-\tau}\right)\varsigma^{\sigma j}+\frac{\partial L}{\partial q_s}\xi_s^{\sigma j}+\frac{\partial L}{\partial\dot q_s}\left(\dot\xi_s^{\sigma j}-\dot q_s\dot\varsigma^{\sigma j}\right)+\dot\varsigma^{\sigma j}+\exp(-L)(t-\tau)^{1-\alpha}\dot G^{\sigma j}-\exp(-L)(t-\tau)^{1-\alpha}Q_s\left(\xi_s^{\sigma(j-1)}-\dot q_s\varsigma^{\sigma(j-1)}\right)=0$$

$$j=0,1,2,\cdots;\sigma=1,2,\cdots,r\tag{15}$$

式中$G^{\sigma j}(\tau,q_s,\dot q_s)$为规范函数，则

$$I_z^{\sigma}=\sum_{j=0}^{z}\nu^j\left\{\exp(L)\cdot(t-\tau)^{\alpha-1}\left[\varsigma^{\sigma j}+\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)\right]+G^{\sigma j}\right\}\quad\sigma=1,2,\cdots,r\tag{16}$$

是该系统的z阶近似Noether不变量。式(15)中约定当$j=0$时$\varsigma^{\sigma(j-1)}=\xi_s^{\sigma(j-1)}=0$。

证明

$$\begin{aligned}\frac{\mathrm d}{\mathrm d\tau}I_z^{\sigma}=&\sum_{j=0}^{z}\nu^j\left\{\exp(L)\cdot(t-\tau)^{\alpha-1}\left[\frac{\mathrm dL}{\mathrm d\tau}\varsigma^{\sigma j}+\frac{\mathrm dL}{\mathrm d\tau}\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)-\frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)-\right.\right.\\&\left.\left.\frac{\alpha-1}{t-\tau}\varsigma^{\sigma j}+\frac{\mathrm d}{\mathrm d\tau}\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\frac{\partial L}{\partial\dot q_s}\left(\dot\xi_s^{\sigma j}-\dot q_s\dot\varsigma^{\sigma j}-\ddot q_s\varsigma^{\sigma j}\right)+\dot\varsigma^{\sigma j}\right]+\dot G^{\sigma j}\right\}=\\&\sum_{j=0}^{z}\nu^j\left\{\exp(L)(t-\tau)^{\alpha-1}\left[\frac{\mathrm dL}{\mathrm d\tau}\varsigma^{\sigma j}+\frac{\partial L}{\partial q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)-\frac{\alpha-1}{t-\tau}\varsigma^{\sigma j}+\dot\varsigma^{\sigma j}+\frac{\partial L}{\partial\dot q_s}\left(\dot\xi_s^{\sigma j}-\dot q_s\dot\varsigma^{\sigma j}-\ddot q_s\varsigma^{\sigma j}\right)\right]-\right.\\&\left.\nu Q_s\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\dot G^{\sigma j}\right\}=\sum_{j=0}^{z}\nu^j\left[Q_s\left(\xi_s^{\sigma(j-1)}-\dot q_s\varsigma^{\sigma(j-1)}\right)-\nu Q_s\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)\right]=-\nu^{z+1}Q_s\left(\xi_s^{\sigma z}-\dot q_s\varsigma^{\sigma z}\right)\end{aligned}\tag{17}$$

故I_z^{σ}是El-Nabulsi指数型动力学系统(5)的z阶近似Noether不变量。

同理，假设在小扰动νQ_s作用下El-Nabulsi幂律型动力学方程(7)成为

$$(1+\gamma)(t-\tau)^{\alpha-1}L^{\gamma}\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm d}{\mathrm d\tau}\frac{\partial L}{\partial\dot q_s}-\frac{\gamma}{L}\frac{\partial L}{\partial\dot q_s}\frac{\mathrm dL}{\mathrm d\tau}+\frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial\dot q_s}\right)=\nu Q_s\tag{18}$$

定理4 已知El-Nabulsi幂律型动力学系统(7)受小扰动的作用，如果生成元$\varsigma^{\sigma j}$、$\xi_s^{\sigma j}$满足结构方程

$$\left(\frac{\partial L}{\partial\tau}+\frac{L}{1+\gamma}\frac{1-\alpha}{t-\tau}\right)\varsigma^{\sigma j}+\frac{\partial L}{\partial q_s}\xi_s^{\sigma j}+\frac{\partial L}{\partial\dot q_s}\left(\dot\xi_s^{\sigma j}-\dot q_s\dot\varsigma^{\sigma j}\right)+\frac{L}{1+\gamma}\dot\varsigma^{\sigma j}+\frac{(t-\tau)^{1-\alpha}}{(1+\gamma)L^{\gamma}}\dot G^{\sigma j}-\frac{(t-\tau)^{1-\alpha}}{(1+\gamma)L^{\gamma}}Q_s\left(\xi_s^{\sigma(j-1)}-\dot q_s\varsigma^{\sigma(j-1)}\right)=0$$

$$j=0,1,2,\cdots;\sigma=1,2,\cdots,r\tag{19}$$

则

$$I_z^{\sigma}=\sum_{j=0}^{z}\nu^j\left\{(t-\tau)^{\alpha-1}L^{\gamma}\left[L\varsigma^{\sigma j}+(1+\gamma)\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)\right]+G^{\sigma j}\right\}\quad\sigma=1,2,\cdots,r\tag{20}$$

是该系统的z阶近似Noether不变量。式(19)中约定，当$j=0$时$\varsigma^{\sigma(j-1)}=\xi_s^{\sigma(j-1)}=0$。

证明

$$\begin{aligned}\frac{\mathrm d}{\mathrm d\tau}I_{Nz}^{\sigma}=&\sum_{j=0}^{z}\nu^j\left\{(t-\tau)^{\alpha-1}L^{\gamma}\left[-\frac{\alpha-1}{t-\tau}L\varsigma^{\sigma j}-(1+\gamma)\frac{\alpha-1}{t-\tau}\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\gamma\frac{\mathrm dL}{\mathrm d\tau}\varsigma^{\sigma j}+\right.\right.\\&(1+\gamma)\frac{\gamma}{L}\frac{\mathrm dL}{\mathrm d\tau}\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\frac{\mathrm dL}{\mathrm d\tau}\varsigma^{\sigma j}+(1+\gamma)\frac{\mathrm d}{\mathrm d\tau}\frac{\partial L}{\partial\dot q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+(1+\gamma)\frac{\partial L}{\partial\dot q_s}\left(\dot\xi_s^{\sigma j}-\dot q_s\dot\varsigma^{\sigma j}-\right.\\&\left.\left.\left.\ddot q_s\varsigma^{\sigma j}\right)+L\dot\varsigma^{\sigma j}\right]+\dot G^{\sigma j}\right\}=\\&\sum_{j=0}^{z}\nu^j\left\{(t-\tau)^{\alpha-1}L^{\gamma}\left[-\frac{\alpha-1}{t-\tau}L\varsigma^{\sigma j}+(1+\gamma)\frac{\partial L}{\partial q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\gamma\frac{\mathrm dL}{\mathrm d\tau}\varsigma^{\sigma j}+\frac{\mathrm dL}{\mathrm d\tau}\varsigma^{\sigma j}+(1+\gamma)\frac{\partial L}{\partial\dot q_s}\left(\dot\xi_s^{\sigma j}-\right.\right.\right.\\&\left.\left.\left.\dot q_s\dot\varsigma^{\sigma j}-\ddot q_s\varsigma^{\sigma j}\right)+L\dot\varsigma^{\sigma j}\right]-\nu Q_s\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\dot G^{\sigma j}\right\}=\\&\sum_{j=0}^{z}\nu^j\left\{(t-\tau)^{\alpha-1}L^{\gamma}(1+\gamma)\left[-\frac{L}{1+\gamma}\frac{\alpha-1}{t-\tau}\varsigma^{\sigma j}+\frac{\partial L}{\partial q_s}\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\frac{\mathrm dL}{\mathrm d\tau}\varsigma^{\sigma j}+\frac{\partial L}{\partial\dot q_s}\left(\dot\xi_s^{\sigma j}-\dot q_s\dot\varsigma^{\sigma j}-\right.\right.\right.\\&\left.\left.\left.\ddot q_s\varsigma^{\sigma j}\right)+\frac{L}{(1+\gamma)}\dot\varsigma^{\sigma j}\right]-\nu Q_s\left(\xi_s^{\sigma j}-\dot q_s\varsigma^{\sigma j}\right)+\dot G^{\sigma j}\right\}=\end{aligned}$$

$$\sum_{j=0}^{z}\nu^j\left[Q_s\left(\xi_s^{\sigma(j-1)}-\dot{q}_s\varsigma^{\sigma(j-1)}\right)-\nu Q_s\left(\xi_s^{\sigma j}-\dot{q}_s\varsigma^{\sigma j}\right)\right]=-\nu^{z+1}Q_s\left(\xi_s^{\sigma z}-\dot{q}_s\varsigma^{\sigma z}\right) \tag{21}$$

故I_z^σ是El-Nabulsi幂律型动力学系统(7)的z阶近似Noether不变量。

近似Noether不变量(16)和(20)是由非保守系统的近似Noether对称性所导致的,它们是在小扰动作用下近似保持不变的守恒量。若未受扰动,则分别退化为精确不变量(10)和(12)。若$I_0=0$,则称近似不变量是平凡的[16]。

4 算　　例

例1　设El-Nabulsi指数型动力学系统的作用量为[4]

$$S_{\mathrm{E}}=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\exp\left(\tau q\dot{q}\right)(t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{22}$$

方程(5)给出

$$(t-\tau)^{\alpha-1}\mathrm{e}^{\tau q\dot{q}}\left[\tau q\left(\frac{\alpha-1}{t-\tau}-q\dot{q}-\tau\dot{q}^2-\tau q\ddot{q}\right)-q\right]=0 \tag{23}$$

这是一个非线性动力学系统。方程(9)给出

$$\left(q\dot{q}+\frac{1-\alpha}{t-\tau}\right)\varsigma^0+\tau\dot{q}\xi^0+\tau q\left(\dot{\xi}^0-\dot{q}\dot{\varsigma}^0\right)+\dot{\varsigma}^0+\mathrm{e}^{-\tau q\dot{q}}(t-\tau)^{1-\alpha}\dot{G}^0=0 \tag{24}$$

方程(24)有解,为

$$\varsigma^0=0,\ \xi^0=\frac{1}{q},\ G^0=0 \tag{25}$$

由定理1得到

$$I_0=\tau\mathrm{e}^{\tau q\dot{q}}(t-\tau)^{\alpha-1}=\mathrm{const} \tag{26}$$

式(26)为系统未受扰动时的不变量,因此是精确Noether不变量。

假设系统受到小扰动

$$\nu Q=-\nu\frac{q}{\tau}\mathrm{e}^{\tau q\dot{q}}(t-\tau)^{\alpha-1} \tag{27}$$

方程(15)给出

$$\left(q\dot{q}+\frac{1-\alpha}{t-\tau}\right)\varsigma^1+\tau\dot{q}\xi^1+\tau q\left(\dot{\xi}^1-\dot{q}\dot{\varsigma}^1\right)+\dot{\varsigma}^1+\mathrm{e}^{-\tau q\dot{q}}(t-\tau)^{1-\alpha}\dot{G}^1-\mathrm{e}^{-\tau q\dot{q}}(t-\tau)^{1-\alpha}Q\left(\xi^0-\dot{q}\varsigma^0\right)=0 \tag{28}$$

方程(28)有解,为

$$\varsigma^1=1,\ \xi^1=\dot{q},\ G^1=0 \tag{29}$$

由定理3得到

$$I_1=\tau\mathrm{e}^{\tau q\dot{q}}(t-\tau)^{\alpha-1}+\nu\mathrm{e}^{\tau q\dot{q}}(t-\tau)^{\alpha-1} \tag{30}$$

这是一阶近似Noether不变量。为求得二阶近似不变量,在方程(15)中取$j=2$,则有

$$\left(q\dot{q}+\frac{1-\alpha}{t-\tau}\right)\varsigma^2+\tau\dot{q}\xi^2+\tau q\left(\dot{\xi}^2-\dot{q}\dot{\varsigma}^2\right)+\dot{\varsigma}^2+\mathrm{e}^{-\tau q\dot{q}}(t-\tau)^{1-\alpha}\dot{G}^2-\mathrm{e}^{-\tau q\dot{q}}(t-\tau)^{1-\alpha}Q\left(\xi^1-\dot{q}\varsigma^1\right)=0 \tag{31}$$

方程(31)有解,为

$$\varsigma^2=0,\ \xi^2=\frac{1}{q},\ G^2=0 \tag{32}$$

由定理3得到

$$I_2=\tau\mathrm{e}^{\tau q\dot{q}}(t-\tau)^{\alpha-1}+\nu\mathrm{e}^{\tau q\dot{q}}(t-\tau)^{\alpha-1}+\nu^2\tau\mathrm{e}^{\tau q\dot{q}}(t-\tau)^{\alpha-1} \tag{33}$$

这是系统的二阶近似Noether不变量。进一步可求得更高阶的近似不变量。

例2　设El-Nabulsi幂律型动力学系统的作用量为[4]

$$S_{\mathrm{P}}=\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left[\dot{q}-q(\tau-t)\right]^2(t-\tau)^{\alpha-1}\mathrm{d}\tau \tag{34}$$

方程(7)给出

$$2(t-\tau)^{\alpha-1}\left\{\left[(t-\tau)^2+\alpha\right]q-\ddot{q}+\frac{\alpha-1}{t-\tau}\dot{q}\right\}=0 \tag{35}$$

这是一个非保守动力学系统。方程(11)给出

$$\left[\frac{\dot{q}}{2}\frac{1-\alpha}{t-\tau}-\frac{q}{2}(1+\alpha)\right]\varsigma^0+(t-\tau)\xi^0+\dot{\xi}^0-\frac{\dot{q}+q(\tau-t)}{2}\dot{\varsigma}^0+\frac{(t-\tau)^{1-\alpha}}{2\left[\dot{q}-q(\tau-t)\right]}G^0=0 \tag{36}$$

方程(36)有解,为

$$\varsigma^0=0,\ \xi^0=\mathrm{e}^{\frac{1}{2}(\tau-t)^2},\ G^0=0 \tag{37}$$

由定理2,系统有精确Noether不变量

$$I_0=2\mathrm{e}^{\frac{1}{2}(\tau-t)^2}\left[\dot{q}-q(\tau-t)\right](t-\tau)^{\alpha-1}=\mathrm{const} \tag{38}$$

假设系统受到小扰动

$$\nu Q=-2\nu\mathrm{e}^{-\frac{1}{2}(\tau-t)^2}\left[\dot{q}-q(\tau-t)\right](t-\tau)^{\alpha} \tag{39}$$

方程(19)给出

$$\left[\frac{\dot{q}}{2}\frac{1-\alpha}{t-\tau}-\frac{q}{2}(1+\alpha)\right]\varsigma^1+(t-\tau)\xi^1+\dot{\xi}^1-\frac{\dot{q}+q(\tau-t)}{2}\dot{\varsigma}^1+\frac{(t-\tau)^{1-\alpha}}{2\left[\dot{q}-q(\tau-t)\right]}\dot{G}^1-\frac{(t-\tau)^{1-\alpha}}{2\left[\dot{q}-q(\tau-t)\right]}Q\left(\xi^0-\dot{q}\varsigma^0\right)=0 \tag{40}$$

方程(40)有解,为

$$\varsigma^1=0,\ \xi^1=-1,\ G^1=0 \tag{41}$$

由定理4得到

$$I_1^\sigma = 2e^{\frac{1}{2}(\tau-t)^2}\left[\dot{q}-q(\tau-t)\right](t-\tau)^{\alpha-1}-2\nu\left[\dot{q}-q(\tau-t)\right](t-\tau)^{\alpha-1} \tag{42}$$

式(42)为一阶近似Noether不变量。进一步可求得更高阶的近似不变量。

5 结 论

El-Nabulsi指数模型和幂律模型为研究非保守系统动力学提供了一个新的思路。本文研究非保守系统在小扰动作用下的近似Noether不变量问题，提出并证明了El-Nabulsi指数模型和El-Nabulsi幂律模型下非保守系统的近似Noether不变量定理，给出了不变量的形式，即式(16)和(20)及其存在条件(15)和(19)。算例表明了结果的有效性。

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