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基于非标准 Lagrange 函数的动力学系统的广义能量积分与 Whittaker 降阶法

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摘要:研究基于非标准 Lagrange 函数的动力学系统的广义能量积分和 Whittaker 降阶方法。首先,基于指数 Lagrange 函数和 Lagrange 函数幂函数两类非标准 Lagrange 函数,定义了相应的 Hamilton 作用量,建立了该系统的 Hamilton 原理,给出了系统的 Lagrange 方程。其次,利用系统的 Lagrange 方程,建立了基于非标准 Lagrange 函数的广义能量积分存在的条件及形式。然后,将著名的 Whittaker 降阶法加以推广,得到了基于非标准 Lagrange 函数的动力学系统的 Whittaker 方程。最后,以算例验证了本文结果。

关键词:非标准 Lagrange 函数;非线性动力学;广义能量积分;Whittaker 降阶法

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Generalized Energy Integral and Whittaker Method of Reduction for Dynamics Systems with Non-standard Lagrangians

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Abstract: The generalized energy integral and Whittaker method of reduction for the dynamics system based on non-standard Lagrangians are studied. Firstly, in view of two kinds of non-standard Lagrangians, i. e., exponential Lagrangians and power law Lagrangians, the Hamilton action with non-standard Lagrangians is defined, and the Hamilton principles and the Lagrange equations of the system are obtained. Secondly, the condition under which the generalized energy integral with non-standard Lagrangians exists and the form of generalized energy integral are established by using the Lagrange equations of the system. Thirdly, the famous Whittaker method of reduction is extended, and the Whittaker equations for the dynamics system with non-standard Lagrangians are obtained. Finally, an example is given to illustrate the application of the results.

Key words: non-standard Lagrangians; nonlinear dynamics; generalized energy integral; Whittaker method of reduction

自然界中最普遍的问题都是关于非保守非线性问题,而非线性问题可用非标准 Lagrange 函数的变分问题来解决,因此研究基于非标准 Lagrange 函数的动力学系统具有重要的理论意义与

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应用价值。事实上，非标准 Lagrange 函数的思想可以追溯至 1978 年 Arnold 的工作[1-2]。但是由于缺乏与之对应的哈密顿形式而被忽视，弦理论[3-5]中被重新考虑。与标准 Lagrange 函数表示为动能和势能之差不同，非标准 Lagrange 函数没有关于动能和势能的明显区分，具有指数形式、对数形式等。这种不规则的 Lagrange 函数凭借其在非线性动力学系统[6-8]、耗散系统[9-12]和量子场理论[13-15]中的重要作用引起了学者们的关注。Musielak[9-10]研究了耗散系统中获得非标准 Lagrange 函数的方法及其存在的条件。El-Nabulsi[15]研究了非线性动力学系统基于两类非标准 Lagrange 函数的作用量及动力学方程。在宇宙学方面，非标准 Lagrange 函数也扮演了重要角色，Dimitrijevic[16]等将非标准 Lagrange 函数考虑进现代宇宙学模型中，研究了其运动方程。EL-Nabulsi[17]应用非标准 Lagrange 函数到 Friedmann-Robertson-Walker 时空，讨论了它在宇宙学中的影响。关于非标准 Lagrange 函数应用的研究还有其他一些重要成果[18-22]，但尚未涉及能量积分及降阶法。

动力学系统的守恒量，表现为深刻的物理规律。分析力学发展后，通过循环积分[23-26]和广义能量积分[27-31]可以寻求系统的守恒量从而对系统约化。1904 年，Whittaker[32]利用能量积分降阶了完整保守系统的运动方程，得到了 Whittaker 方程。之后，Whittaker 降阶法引起了学者们的关注。然而，目前为止，对 Whittaker 方程的研究限于标准 Lagrange 函数系统。

本文将研究动力学系统基于指数 Lagrange 函数和 Lagrange 函数幂函数的广义能量积分及降阶方法。通过变分方法得到系统的 Lagrange 方程，给出系统广义能量积分存在的条件及形式，建立基于两类非标准 Lagrange 函数的动力学系统的 Whittaker 方法。

1 基于指数 Lagrange 函数的动力学系统的广义能量积分与 Whittaker 降阶法

1.1 Lagrange 方程

假设系统的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 确定，则基于指数 Lagrange 函数的作用量定义为[15]

$$S=\int_{t_1}^{t_2}\exp[L(t,q_s,\dot{q}_s)]\mathrm{d}t \tag{1}$$

式中：$L=L(t,q_s,\dot{q}_s)$ 为经典意义下的 Lagrange 函数。与基于指数 Lagrange 函数的作用量式(1)相应的 Hamilton 原理可写为

$$\delta S=0 \tag{2}$$

带有交换关系

$$d\delta q_s=\delta \mathrm{d}q_s \quad s=1,2,\cdots,n \tag{3}$$

以及边界条件

$$\delta q_s|_{t=t_1}=\delta q_s|_{t=t_2}=0 \qquad s=1,2,\cdots,n \tag{4}$$

因为

$$\begin{aligned}\delta\exp(L)=&\sum_{s=1}^{n}\exp(L)\frac{\partial L}{\partial q_s}\delta q_s+\sum_{s=1}^{n}\exp(L)\frac{\partial L}{\partial \dot{q}_s}\delta\dot{q}_s=\\&\sum_{s=1}^{n}\exp(L)\frac{\partial L}{\partial q_s}\delta q_s+\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{s=1}^{n}\exp(L)\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right)-\\&\sum_{s=1}^{n}\exp(L)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right)\delta q_s-\sum_{s=1}^{n}\exp(L)\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\delta q_s\end{aligned} \tag{5}$$

将式(5)代入式(2)，有

$$\int_{t_1}^{t_2}\sum_{s=1}^{n}\left[\begin{aligned}&\exp(L)\frac{\partial L}{\partial q_s}-\exp(L)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-\\&\quad\exp(L)\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\end{aligned}\right]\delta q_s\mathrm{d}t+\sum_{s=1}^{n}\exp(L)\frac{\partial L}{\partial \dot{q}_s}\delta q_s\Big|_{t_1}^{t_2}=0 \tag{6}$$

利用边界条件式(4)，得到

$$\int_{t_1}^{t_2}\sum_{s=1}^{n}\left[\begin{aligned}&\exp(L)\frac{\partial L}{\partial q_s}-\exp(L)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-\\&\quad\exp(L)\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\end{aligned}\right]\delta q_s\mathrm{d}t=0 \tag{7}$$

对于完整系统，$\delta q_s(s=1,2,\cdots,n)$ 相互独立，由变分学基本引理[33]，得到

$$\exp(L)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\right)=0 \qquad s=1,2,\cdots,n \tag{8}$$

方程式(8)称为基于指数 Lagrange 函数的动力学系统的 Lagrange 方程[15]。

1.2 广义能量积分

为了得到所论系统的广义能量积分，计算

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\sum_{s=1}^{n}\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}-1\right)\exp(L)\right]=\\&\sum_{s=1}^{n}\exp(L)\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}+\exp(L)\ddot{q}_s\frac{\partial L}{\partial \dot{q}_s}+\\&\exp(L)\dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\exp(L)\frac{\mathrm{d}L}{\mathrm{d}t}=\\&\sum_{s=1}^{n}\exp(L)\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}+\exp(L)\ddot{q}_s\frac{\partial L}{\partial \dot{q}_s}+\\&\exp(L)\dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\\&\exp(L)\left(\frac{\partial L}{\partial t}+\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s\right)=\\&\sum_{s=1}^{n}\left[\left(\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}\right)\dot{q}_s-\frac{\partial L}{\partial t}\right]\exp(L)\end{aligned} \tag{9}$$

由方程式(8,9)知，如果 Lagrange 函数不显含时间 t，即

$$\frac{\partial L}{\partial t}=0 \tag{10}$$

则沿着系统的动力学轨线有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\sum_{s=1}^{n}\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}-1\right)\exp(L)\right]=0 \tag{11}$$

于是系统存在广义能量积分

$$\left(\sum_{s=1}^{n}\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}-1\right)\exp(L)=h=\mathrm{const} \tag{12}$$

1.3　Whittaker 降阶法

任取一个广义坐标代替 t 的作用，如取 q_1，令

$$q'_r=\frac{\mathrm{d}q_r}{\mathrm{d}q_1}\quad r=2,3,\cdots,n \tag{13}$$

则有

$$\dot{q}_r=\frac{\mathrm{d}q_r}{\mathrm{d}q_1}\frac{\mathrm{d}q_1}{\mathrm{d}t}=q'_r\dot{q}_1\quad r=2,3,\cdots,n \tag{14}$$

设

$$\exp L^*(\dot{q}_1,q'_r,q_s)=\exp L(\dot{q}_1,q'_r\dot{q}_1,q_s) \tag{15}$$

将式(15)对 $\dot{q}_1,q'_r,q_s$ 分别求偏导数，得

$$\exp(L^*)\frac{\partial L^*}{\partial \dot{q}_1}=\exp(L)\left(\frac{\partial L}{\partial \dot{q}_1}+\sum_{r=2}^{n}\frac{\partial L}{\partial \dot{q}_r}\frac{\partial \dot{q}_r}{\partial \dot{q}_1}\right)=\exp(L)\left(\frac{\partial L}{\partial \dot{q}_1}+\sum_{r=2}^{n}\frac{\partial L}{\partial \dot{q}_r}\frac{\dot{q}_r}{\dot{q}_1}\right)=\sum_{s=1}^{n}\exp(L)\frac{\partial L}{\partial \dot{q}_s}\frac{\dot{q}_s}{\dot{q}_1} \tag{16}$$

$$\exp(L^*)\frac{\partial L^*}{\partial q'_r}=\exp(L)\frac{\partial L}{\partial \dot{q}_r}\dot{q}_1\quad r=2,3,\cdots,n \tag{17}$$

$$\exp(L^*)\frac{\partial L^*}{\partial q_s}=\exp(L)\frac{\partial L}{\partial q_s}\quad s=1,2,\cdots,n \tag{18}$$

利用广义能量积分式(12,14)解出 $\dot{q}_1$，记作

$$\dot{q}_1=\dot{q}_1(q'_r,q_s) \tag{19}$$

根据式(15,16)，式(12)可表示成

$$\exp(L^*)\frac{\partial L^*}{\partial \dot{q}_1}\dot{q}_1-\exp(L^*)=h \tag{20}$$

其中 $\dot{q}_1$ 由式(19)确定，将式(20)分别对 q'_r 和 q_s 求偏导数，有

$$\exp(L^*)\left(\frac{\partial L^*}{\partial q'_r}\frac{\partial L^*}{\partial \dot{q}_1}+\frac{\partial^2 L^*}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q'_r}+\frac{\partial^2 L^*}{\partial \dot{q}_1\partial q'_r}\right)\dot{q}_1-\exp(L^*)\frac{\partial L^*}{\partial q'_r}=0 \tag{21}$$

$$\exp(L^*)\left(\frac{\partial L^*}{\partial q_s}\frac{\partial L^*}{\partial \dot{q}_1}+\frac{\partial^2 L^*}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_s}+\frac{\partial^2 L^*}{\partial \dot{q}_1\partial q_s}\right)\dot{q}_1-\exp(L^*)\frac{\partial L^*}{\partial q_s}=0 \tag{22}$$

令

$$\exp W(q'_r,q_s)=\exp(L^*)\frac{\partial L^*}{\partial \dot{q}_1} \tag{23}$$

将式(23)对 q'_r,q_s 求偏微分，得

$$\exp(W)\frac{\partial W}{\partial q'_r}=\exp(L^*)\left(\frac{\partial L^*}{\partial q'_r}\frac{\partial L^*}{\partial \dot{q}_1}+\frac{\partial^2 L^*}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q'_r}+\frac{\partial^2 L^*}{\partial \dot{q}_1\partial q'_r}\right) \tag{24}$$

$$\exp(W)\frac{\partial W}{\partial q_s}=\exp(L^*)\left(\frac{\partial L^*}{\partial q_s}\frac{\partial L^*}{\partial \dot{q}_1}+\frac{\partial^2 L^*}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_s}+\frac{\partial^2 L^*}{\partial \dot{q}_1\partial q_s}\right) \tag{25}$$

由式(21,24)，得

$$\exp(W)\frac{\partial W}{\partial q'_r}=\frac{1}{\dot{q}_1}\exp(L^*)\frac{\partial L^*}{\partial q'_r} \tag{26}$$

由式(22,25)，得

$$\exp(W)\frac{\partial W}{\partial q_s}=\frac{1}{\dot{q}_1}\exp(L^*)\frac{\partial L^*}{\partial q_s} \tag{27}$$

再根据式(17,18)，得

$$\exp(W)\frac{\partial W}{\partial q'_r}=\exp(L)\frac{\partial L}{\partial \dot{q}_r}\quad r=2,3,\cdots,n \tag{28}$$

$$\exp(W)\frac{\partial W}{\partial q_s}=\frac{1}{\dot{q}_1}\exp(L)\frac{\partial L}{\partial q_s}\quad s=1,2,\cdots,n \tag{29}$$

方程式(8)可表示成

$$\exp(L)\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(L)\frac{\partial L}{\partial \dot{q}_s}\right)=0 \tag{30}$$

将式(28,29)代入式(30)，有

$$\exp(W)\frac{\partial W}{\partial q_r}\dot{q}_1-\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(W)\frac{\partial W}{\partial q'_r}\right)=0$$

或

$$\exp(W)\frac{\partial W}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}q_1}\left(\exp(W)\frac{\partial W}{\partial q'_r}\right)=0 \tag{31}$$

展开之后，得到 Lagrange 函数的 Whittaker 方程

$$\exp(W)\left(\frac{\partial W}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}q_1}\frac{\partial W}{\partial q'_r}-\frac{\partial W}{\partial q'_r}\frac{\mathrm{d}W}{\mathrm{d}q_1}\right)=0\quad r=2,3,\cdots,n \tag{32}$$

式中：W 为关于 $q'_2,q'_3,\cdots,q'_n,q_1,q_2,\cdots,q_n$ 的函数，而 q_1 是相当于时间 t 的独立变量。Whittaker 方程的形式等同于运动方程式(8)，但方程个数变成 $n-1$ 个，从而达到了降阶的目的。

1.4　算例

设基于指数 Lagrange 函数的作用量为

$$S=\int_{t_1}^{t_2}e^{-q_1q_2}(\dot{q}_1^2+\dot{q}_2^2)\,\mathrm{d}t \tag{33}$$

试求广义能量积分，并用其将方程降阶。

本问题中，经典的 Lagrange 函数为

$$L=\ln(\dot{q}_1^2+\dot{q}_2^2)-q_1q_2 \tag{34}$$

由方程式(8)得到

$$\begin{aligned}&2\ddot{q}_1-\dot{q}_1^2q_2-2\dot{q}_1\dot{q}_2q_1+\dot{q}_2^2q_2=0\\&2\ddot{q}_2-\dot{q}_2^2q_1-2\dot{q}_2\dot{q}_1q_2+\dot{q}_1^2q_1=0\end{aligned} \tag{35}$$

方程式(35)为非线性微分方程组。

由式(34)知

$$\frac{\partial L}{\partial t}=0 \tag{36}$$

因此存在能量积分

$$(\dot{q}_1^2+\dot{q}_2^2)e^{-q_1q_2}=h \tag{37}$$

令 $\dot{q}_2=\dot{q}_1q'_2$ 代入式(37),有

$$\dot{q}_1^2(1+q'^2_2)e^{-q_1q_2}=h \tag{38}$$

解出 $\dot{q}_1$,得

$$\dot{q}_1=\sqrt{\frac{he^{q_1q_2}}{1+q'^2_2}} \tag{39}$$

根据式(15)构造 $\exp L^*$ 函数,有

$$\exp L^*=\exp(\ln(\dot{q}_1^2(1+q'^2_2))-q_1q_2)=\dot{q}_1^2(1+q'^2_2)e^{-q_1q_2} \tag{40}$$

构造 $\exp W$ 函数

$$\exp W=\exp(L^*)\frac{\partial L^*}{\partial \dot{q}_1}=2\dot{q}_1(1+q'^2_2)e^{-q_1q_2} \tag{41}$$

将式(39)代入式(41),整理后得

$$\exp W=2\sqrt{he^{-q_1q_2}(1+q'^2_2)} \tag{42}$$

由方程式(32),得

$$\exp(W)\left(\frac{\partial W}{\partial q_2}-\frac{\mathrm{d}}{\mathrm{d}q_1}\frac{\partial W}{\partial q'_2}-\frac{\partial W}{\partial q'_2}\frac{\mathrm{d}W}{\mathrm{d}q_1}\right)=0 \tag{43}$$

即

$$q_2q'_2-q_1q'^2_2-q_1=0 \tag{44}$$

2　基于 Lagrange 函数幂函数的动力学系统的广义能量积分与 Whittaker 降阶法

2.1　Lagrange 方程

假设系统的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 确定,则基于 Lagrange 函数幂函数的作用量定义为[15]

$$S=\int_{t_1}^{t_2}L^{1+\gamma}(t,q_s,\dot{q}_s)\,\mathrm{d}t \tag{45}$$

式中:$L=L(t,q_s,\dot{q}_s)$是经典意义下的 Lagrange 函数,γ为任意实数。与基于 Lagrange 函数幂函数的作用量(式(45))相应的 Hamilton 原理可写为

$$\delta S=0 \tag{46}$$

带有交换关系

$$\mathrm{d}\delta q_s=\delta\mathrm{d}q_s\qquad s=1,2,\cdots,n \tag{47}$$

以及边界条件

$$\delta q_s|_{t=t_1}=\delta q_s|_{t=t_2}=0\qquad s=1,2,\cdots,n \tag{48}$$

因为

$$\delta L^{1+\gamma}=(1+\gamma)L^\gamma\left(\sum_{s=1}^n\frac{\partial L}{\partial q_s}\delta q_s+\sum_{s=1}^n\frac{\partial L}{\partial \dot{q}_s}\delta\dot{q}_s\right)=(1+\gamma)L^\gamma\sum_{s=1}^n\frac{\partial L}{\partial q_s}\delta q_s+\frac{\mathrm{d}}{\mathrm{d}t}\left[(1+\gamma)L^\gamma\sum_{s=1}^n\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right]-(1+\gamma)L^\gamma\sum_{s=1}^n\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}_s}\right)+\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\right]\delta q_s \tag{49}$$

将式(49)代入式(46),有

$$\int_{t_1}^{t_2}(1+\gamma)L^\gamma\sum_{s=1}^n\left[\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\right]\delta q_s\cdot\mathrm{d}t+(1+\gamma)L^\gamma\frac{\partial L}{\partial \dot{q}_s}\delta q_s\bigg|_{t_1}^{t_2}=0 \tag{50}$$

利用边界条件式(48),得到

$$\int_{t_1}^{t_2}(1+\gamma)L^\gamma\sum_{s=1}^n\left[\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_s}\right)-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\right]\cdot\delta q_s\mathrm{d}t=0 \tag{51}$$

对于完整系统,$\delta q_s(s=1,2,\cdots,n)$相互独立,由变分学基本引理[33],得到

$$(1+\gamma)L^\gamma\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}\right)=0\qquad s=1,2,\cdots,n \tag{52}$$

当 $\gamma\neq-1$ 时,有

$$\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}=0\qquad s=1,2,\cdots,n \tag{53}$$

方程式(53)称为基于 Lagrange 函数幂函数的动力学系统的 Lagrange 方程[15]。

若 $\gamma=0$,方程式(52)为经典的 Lagrange 运动方程,即

$$\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}=0\qquad s=1,2,\cdots,n \tag{54}$$

2.2　广义能量积分

下面计算系统的广义能量积分。因为

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\sum_{s=1}^n(1+\gamma)\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}-L\right)L^\gamma\right]=\\&\sum_{s=1}^n(1+\gamma)L^\gamma\left(\ddot{q}_s\frac{\partial L}{\partial \dot{q}_s}+\frac{\gamma}{L}\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}+\dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}\right)-\\&(1+\gamma)L^\gamma\frac{\mathrm{d}L}{\mathrm{d}t}-\\&\sum_{s=1}^n(1+\gamma)L^\gamma\left(\ddot{q}_s\frac{\partial L}{\partial \dot{q}_s}+\frac{\gamma}{L}\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}+\dot{q}_s\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}\right)-\\&(1+\gamma)L^\gamma\left(\frac{\partial L}{\partial t}+\frac{\partial L}{\partial q_s}\dot{q}_s+\frac{\partial L}{\partial \dot{q}_s}\ddot{q}_s\right)=\\&\sum_{s=1}^n(1+\gamma)L^\gamma\cdot\\&\left[\left(\frac{\gamma}{L}\frac{\partial L}{\partial \dot{q}_s}\frac{\mathrm{d}L}{\mathrm{d}t}+\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}\right)\dot{q}_s-\frac{\partial L}{\partial t}\right]\end{aligned} \tag{55}$$

由方程式(53,55)知,如果 Lagrange 函数不显含时间 t,即

$$\frac{\partial L}{\partial t}=0 \tag{56}$$

则沿着系统的动力学轨线有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\sum_{s=1}^{n}(1+\gamma)\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}-L\right)L^{\gamma}\right]=0 \tag{57}$$

于是系统存在广义能量积分

$$\left(\sum_{s=1}^{n}(1+\gamma)\dot{q}_s\frac{\partial L}{\partial \dot{q}_s}-L\right)L^{\gamma}=h=\mathrm{const} \tag{58}$$

2.3　Whittaker 降阶法

取广义坐标 q_1 代替时间 t，令

$$q_r'=\frac{\mathrm{d}q_r}{\mathrm{d}q_1}\qquad r=2,3,\cdots,n \tag{59}$$

则有

$$\dot{q}_r=\frac{\mathrm{d}q_r}{\mathrm{d}q_1}\frac{\mathrm{d}q_1}{\mathrm{d}t}=q_r'\dot{q}_1\qquad r=2,3,\cdots,n \tag{60}$$

设

$$\bar{L}^{1+\gamma}(\dot{q}_1,q_r',q_s)=L^{1+\gamma}(\dot{q}_1,q_r'\dot{q}_1,q_s) \tag{61}$$

将式(61)对 $\dot{q}_1,q_r',q_s$ 分别求偏导数，得

$$(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial \dot{q}_1}=(1+\gamma)L^{\gamma}\left(\frac{\partial L}{\partial \dot{q}_1}+\sum_{r=2}^{n}\frac{\partial L}{\partial \dot{q}_r}\frac{\partial \dot{q}_r}{\partial \dot{q}_1}\right)=$$

$$(1+\gamma)L^{\gamma}\left(\frac{\partial L}{\partial \dot{q}_1}+\sum_{r=2}^{n}\frac{\partial L}{\partial \dot{q}_r}\frac{\dot{q}_r}{\dot{q}_1}\right)=$$

$$\sum_{s=1}^{n}(1+\gamma)L^{\gamma}\frac{\partial L}{\partial \dot{q}_s}\frac{\dot{q}_s}{\dot{q}_1} \tag{62}$$

$$(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial q_r'}=(1+\gamma)L^{\gamma}\frac{\partial L}{\partial \dot{q}_r}\dot{q}_1\qquad r=2,3,\cdots,n \tag{63}$$

$$(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial q_s}=(1+\gamma)L^{\gamma}\frac{\partial L}{\partial q_s}\qquad s=1,2,\cdots,n \tag{64}$$

利用广义能量积分式(58,60)解出 $\dot{q}_1$，记作

$$\dot{q}_1=\dot{q}_1(q_r',q_s) \tag{65}$$

根据式(61,62)，式(58)可表示成

$$(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial \dot{q}_1}\dot{q}_1-\bar{L}^{1+\gamma}=h \tag{66}$$

式中 $\dot{q}_1$ 由式(65)确定，将式(66)分别对 q_r' 和 q_s 求偏导数，有

$$(1+\gamma)\bar{L}^{\gamma}\left(\frac{\gamma}{\bar{L}}\frac{\partial \bar{L}}{\partial q_r'}\frac{\partial \bar{L}}{\partial \dot{q}_1}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_r'}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1\partial q_r'}\right)\dot{q}_1-(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial q_r'}=0 \tag{67}$$

$$(1+\gamma)\bar{L}^{\gamma}\left(\frac{\gamma}{\bar{L}}\frac{\partial \bar{L}}{\partial q_s}\frac{\partial \bar{L}}{\partial \dot{q}_1}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_s}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1\partial q_s}\right)\dot{q}_1-(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial q_s'}=0 \tag{68}$$

令

$$\tilde{L}^{1+\gamma}(q_r',q_s)=(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial \dot{q}_1} \tag{69}$$

将式(69)对 q_r',q_s 求偏微分，得

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_r'}=(1+\gamma)\bar{L}^{\gamma}\cdot\left(\frac{\gamma}{L}\frac{\partial \bar{L}}{\partial q_r'}\frac{\partial \bar{L}}{\partial \dot{q}_1}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_r'}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1\partial q_r'}\right) \tag{70}$$

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_s}=(1+\gamma)\bar{L}^{\gamma}\cdot\left(\frac{\gamma}{L}\frac{\partial \bar{L}}{\partial q_s}\frac{\partial \bar{L}}{\partial \dot{q}_1}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1^2}\frac{\partial \dot{q}_1}{\partial q_s}+\frac{\partial^2 \bar{L}}{\partial \dot{q}_1\partial q_s}\right) \tag{71}$$

由式(67,70)，得

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_r'}=\frac{1}{\dot{q}_1}(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial q_r'} \tag{72}$$

由式(68,71)，得

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_s}=\frac{1}{\dot{q}_1}(1+\gamma)\bar{L}^{\gamma}\frac{\partial \bar{L}}{\partial q_s} \tag{73}$$

再根据式(63,64)，得

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_r'}=(1+\gamma)L^{\gamma}\frac{\partial L}{\partial q_r'}\qquad r=2,3,\cdots,n \tag{74}$$

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_s}=\frac{1}{\dot{q}_1}(1+\gamma)L^{\gamma}\frac{\partial L}{\partial q_s}\qquad s=1,2,\cdots,n \tag{75}$$

方程式(52)可表示成

$$(1+\gamma)L^{\gamma}\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\left((1+\gamma)L^{\gamma}\frac{\partial L}{\partial \dot{q}_s}\right)=0 \tag{76}$$

将式(74,75)代入式(76)，有

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_r}\dot{q}_1-\frac{\mathrm{d}}{\mathrm{d}t}\left((1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_r'}\right)=0$$

或

$$(1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}q_1}\left((1+\gamma)\tilde{L}^{\gamma}\frac{\partial \tilde{L}}{\partial q_r'}\right)=0 \tag{77}$$

整理之后，得到

$$\frac{\partial \tilde{L}}{\partial q_r}-\frac{\mathrm{d}}{\mathrm{d}q_1}\frac{\partial \tilde{L}}{\partial q_r'}-\frac{\gamma}{\tilde{L}}\frac{\partial \tilde{L}}{\partial q_r'}\frac{\mathrm{d}\tilde{L}}{\mathrm{d}q_1}=0\qquad r=2,3,\cdots,n \tag{78}$$

方程式(78)称为基于 Lagrange 函数幂函数的动力学系统的 Whittaker 方程。它与方程式(53)相似，只是时间 t 的地位被广义坐标 q_1 取代，但将原 n 个自由度问题降到了 $n-1$ 个自由度问题。

2.4　算例

设基于 Lagrange 函数幂函数的作用量为

$$S=\int_{t_1}^{t_2}[\dot{q}_2(1+q_1^2)+\dot{q}_1+q_2]^{1+\gamma}\mathrm{d}t \tag{79}$$

试求广义能量积分，并用其将方程降阶。

本问题中，经典的 Lagrange 函数为

$$L=\dot{q}_2(1+q_1^2)+\dot{q}_1+q_2 \tag{80}$$

若取 $\gamma=1$，由方程式(53)得到

$$\begin{gathered}2\dot{q}_2q_1(\dot{q}_2(1+q_1^2)+q_2)-(\ddot{q}_2(1+q_1^2)+\ddot{q}_1+\dot{q}_2)=0\\(1-2\dot{q}_1q_1)(\dot{q}_2(1+q_1^2)+\dot{q}_1+q_2)-(1+q_1^2)(\ddot{q}_2(1+q_1^2)+2\dot{q}_2\dot{q}_1q_1+\ddot{q}_1+\dot{q}_2)=0\end{gathered} \tag{81}$$

运动方程式(81)为非线性微分方程组。

由式(80)知

$$\frac{\partial L}{\partial t}=0 \tag{82}$$

则存在广义能量积分

$$[\dot{q}_2(1+q_1^2)+\dot{q}_1]^2-q_2^2=h \tag{83}$$

令 $\dot{q}_2=\dot{q}_1 q'_2$ 代入式(83),有

$$\dot{q}_1^2[q'_2(1+q_1^2)+1]^2-q_2^2=h \tag{84}$$

解出 $\dot{q}_1$,得

$$\dot{q}_1=\frac{\sqrt{h+q_2^2}}{q'_2(1+q_1^2)+1} \tag{85}$$

根据式(61)构造 $\bar{L}$ 函数,有

$$\bar{L}=\dot{q}_1 q'_2(1+q_1^2)+\dot{q}_1+q_2=\dot{q}_1[q'_2(1+q_1^2)+1]+q_2 \tag{86}$$

构造 $\tilde{L}^2$ 函数

$$\tilde{L}^2=(1+\gamma)\bar{L}^\gamma\frac{\partial\bar{L}}{\partial\dot{q}_1}=2\left\{\dot{q}_1[q'_2(1+q_1^2)+1]+q_2\right\}[q'_2(1+q_1^2)+1] \tag{87}$$

将式(85)代入式(87),整理后得

$$\tilde{L}^2=2(\sqrt{h+q_2^2}+q_2)[q'_2(1+q_1^2)+1] \tag{88}$$

由方程式(78),得

$$\frac{\partial\tilde{L}}{\partial q_2}-\frac{\mathrm{d}}{\mathrm{d}q_1}\frac{\partial\tilde{L}}{\partial q'_2}-\frac{1}{\tilde{L}}\frac{\partial\tilde{L}}{\partial q'_2}\frac{\mathrm{d}\tilde{L}}{\mathrm{d}q_1}=0 \tag{89}$$

即

$$[q'_2(1+q_1^2)+1]\left(1+\frac{2q_2}{\sqrt{h+q_2^2}}\right)-2q_1(\sqrt{h+q_2^2}+q_2)=0 \tag{90}$$

3 结束语

利用非标准 Lagrange 函数可以解决非线性问题等,算例中通过 Whittaker 降阶法分别将方程式(35,81)降阶为方程式(44,90),显示了基于非标准 Lagrange 函数的 Whittaker 降阶法在非线性动力学中的优势。文章研究了基于两类非标准 Lagrange 函数的动力学系统的广义能量积分与降阶问题,利用系统的 Lagrange 方程得到了广义能量积分存在的约束条件,并将著名的 Whittaker 降阶法推广到基于非标准 Lagrange 函数的动力学系统,得到了基于非标准 Lagrange 函数的 Whittaker 方程。本文的方法具有普遍意义,可进一步推广到其他非标准 Lagrange 函数系统。同时,也可研究基于非标准 Lagrange 函数的循环积分及 Routh 降阶法。

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