第44卷第2期 南京航空航天大学学报 Vol. 44 No. 2
2012年4月 Journal of Nanjing University of Aeronautics & Astronautics Apr. 2012

广义经典力学系统动力学方程积分的Jacobi最终乘子法

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摘要:研究动力学系统的积分问题,将Jacobi最终乘子法应用于积分广义经典力学系统的动力学方程。建立了广义经典力学系统的运动微分方程;定义了广义经典力学系统的Jacobi最终乘子;研究了系统的第一积分与Jacobi最终乘子的关系。研究表明:对于位形由n个广义坐标确定且拉格朗日函数含有广义坐标对时间的ω阶导数的广义经典力学系统,如果已知系统的$(2\omega n-1)$个第一积分,则可利用Jacobi最终乘子给出系统的解。文末举例说明结果的应用。

关键词:动力学与控制;广义经典力学;积分方法;Jacobi最终乘子

中图分类号:O313 **文献标识码**:A **文章编号**:1005-2615(2012)02-0262-04

Method of Jacobi Last Multiplier for Solving Dynamics Equations Integration of Generalized Classical Mechanics System

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Abstract: The integration issues of dynamic system is studied, and the method of Jacobi last multiplier is applied to integrate dynamic equations of generalized classical mechanics systems. The differential equations of motion of a generalized classical mechanics system are given. The Jacobi last multiplier of the system is defined, and the relation between the Jacobi last multiplier and the first integrals of the system is discussed. The research shows that for a generalized classical mechanics system, whose configuration is determined by n generalized coordinates and Lagrangian contains ω-order derivatives of generalized coordinates with respect to time, the solution of the system can be found by the Jacobi last multiplier if $(2\omega n-1)$ first integrals of the system are known. Finally, an example is given to illustrate the application of the results.

Key words: dynamics and control; generalized classical mechanics; method of integration; Jacobi last multiplier

描述动力学系统的Lagrange函数含广义坐标对时间的高阶导数,简称广义经典力学系统或高阶微商系统。Podolsky[1]指出关于广义经典力学系统的研究可以追溯到Ostrogradsky和Jacobi的工作,文献[2]用现代数学方法对广义经典力学系统和场论进行了系统的描述。对高阶微商系统的研究,特别是对含高阶微商的奇异系统的研究,在引力理论、规范理论、粒子的相对论性动力学、超对称和弦理论等众多领域都是十分重要的[3]。近20年来,在广义经典力学系统的基本理论研究方面已经取得了一系列重要成果[3-13]。本文进一步研究广义经典力学系统的积分方法,将Jocobi提出的研究微分方程积分的最终乘子法[14]推广应用于广义经典力学系统,并举例说明结果的应用。

基金项目:国家自然科学基金(10972151)资助项目。

收稿日期:2011-02-25;**修订日期**:2011-10-07

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1　系统的运动微分方程

研究广义经典力学系统,其位形由 n 个广义坐标 $q^i(i=1,\cdots,n)$ 来确定。系统的 Lagrange 函数为

$$L=L(t,q^i_{(0)},q^i_{(1)},\cdots,q^i_{(\omega)})\quad i=1,\cdots,n \tag{1}$$

式中

$$q^i_{(j)}=\frac{\mathrm{d}^j}{\mathrm{d}t^j}q^i(t)\quad i=1,\cdots,n;j=0,1,\cdots,\omega \tag{2}$$

系统的广义 Euler-Lagrange 方程为[2]

$$\sum_{j=0}^{\omega}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial q^i_{(j)}}=0\quad i=1,\cdots,n \tag{3}$$

引进广义动量和广义 Hamilton 函数为

$$p_i^{(s)}=\sum_{j=0}^{\omega-s-1}(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\partial L}{\partial q^i_{(j+s+1)}}$$
$$i=1,\cdots,n;\ s=0,1,\cdots,\omega-1 \tag{4}$$

$$H(t,\boldsymbol{q}_{(s)},\boldsymbol{p}^{(s)})=p_i^{(s)}\dot{q}^i_{(s)}-L$$
$$i=1,\cdots,n;s=0,1,\cdots,\omega-1 \tag{5}$$

则系统的运动微分方程(3)可表示为正则形式[3]

$$\dot{q}^i_{(s)}=\frac{\partial H}{\partial p_i^{(s)}}\quad \dot{p}_i^{(s)}=-\frac{\partial H}{\partial q^i_{(s)}}$$
$$i=1,\cdots,n;\ s=0,1,\cdots,\omega-1 \tag{6}$$

令

$$a^\mu=\begin{cases}q^i_{(s)} & \mu=2sn+i;s=0,1,\cdots,\omega-1;i=1,\cdots,n\\ p_i^{(s)} & \mu=(2s+1)n+i;s=0,1,\cdots,\omega-1;i=1,\cdots,n\end{cases} \tag{7}$$

$$R_\mu=\begin{cases}p_i^{(s)} & \mu=2sn+i;s=0,1,\cdots,\omega-1;i=1,\cdots,n\\ 0 & \mu=(2s+1)n+i;s=0,1,\cdots,\omega-1;i=1,\cdots,n\end{cases} \tag{8}$$

则方程(6)可进一步表示为逆变代数形式[7]

$$\dot{a}^\mu-\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0\quad \mu=1,\cdots,2\omega n \tag{9}$$

式中

$$\Omega_{\mu\nu}=\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\quad (\Omega_{\mu\nu})=(\Omega^{\mu\nu})^{-1}$$

$$(\Omega^{\mu\nu})=\begin{bmatrix}\boldsymbol{\Omega}_0 & & \\ & \ddots & \\ & & \boldsymbol{\Omega}_{\omega-1}\end{bmatrix}_{2\omega n\times 2\omega n}$$

$$\boldsymbol{\Omega}_0=\cdots=\boldsymbol{\Omega}_{\omega-1}=\begin{bmatrix}\boldsymbol{0}_{n\times n} & \boldsymbol{I}_{n\times n}\\ -\boldsymbol{I}_{n\times n} & \boldsymbol{0}_{n\times n}\end{bmatrix} \tag{10}$$

式中:$\boldsymbol{0}_{n\times n}$为 n 阶零矩阵;$\boldsymbol{I}_{n\times n}$为 n 阶单位矩阵。

2　系统的积分与 Jacobi 最终乘子

引入广义 Poisson 括号

$$[A,H]\overset{\text{def}}{=}\frac{\partial A}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} \tag{11}$$

式中 $A=A(t,\boldsymbol{a})$,括号$[A,H]$满足Lie 代数公理[4]。对于任意可微函数 $I=I(t,\boldsymbol{a})$,沿着广义经典力学系统(9)的运动轨线对时间求导,有

$$\frac{\bar{\mathrm{d}}I}{\mathrm{d}t}=\frac{\partial I}{\partial t}+[I,H] \tag{12}$$

式中

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\frac{\partial}{\partial a^\mu} \tag{13}$$

由式(12),有以下命题。

命题 1[4]　对于广义经典力学系统(9),如果广义 Hamilton 函数 H 不显含时间 t,则 $H=C$ 是系统的第一积分。

命题 2　对于广义经典力学系统(9),$I=I(t,\boldsymbol{a})$是系统的第一积分的充分必要条件是

$$\frac{\partial I}{\partial t}+[I,H]=0 \tag{14}$$

1844 年,Jacobi 在研究微分方程的积分时提出了最终乘子(Last Multiplier)的概念[14],梅凤翔将其应用于积分广义 Hamilton 系统[15]。下面笔者进一步将 Jacobi 最终乘子方法推广应用于广义经典力学系统。

定义　对于广义经典力学系统(9),如果存在函数 $\lambda=\lambda(t,\boldsymbol{a})$满足条件

$$\frac{\partial\lambda}{\partial t}+\frac{\partial}{\partial a^\mu}\left(\lambda\,\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right)=0 \tag{15}$$

则称λ 为系统的Jacobi 最终乘子。方程(15)称为Jacobi 最终乘子的确定方程。

由上述定义,有以下命题。

命题3　对于广义经典力学系统(9),如果满足条件

$$\frac{\partial}{\partial a^\mu}\left(\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right)=0\quad \mu,\nu=1,\cdots,2\omega n \tag{16}$$

则 $\lambda=1$ 是系统的最终乘子,且系统所有的第一积分都是其Jacobi 最终乘子。

命题4　对于广义经典力学系统(9),如果$\lambda=\lambda(t,\boldsymbol{a})$和$\beta=\beta(t,\boldsymbol{a})$是其两个Jacobi 最终乘子,则商$\lambda/\beta$ 是系统的一个第一积分。

证明　由于 $\lambda=\lambda(t,\boldsymbol{a})$和 $\beta=\beta(t,\boldsymbol{a})$是所论广义经典力学系统的两个 Jacobi 最终乘子,根据其定义式(15),有

$$\frac{\partial\lambda}{\partial t}+\frac{\partial}{\partial a^\mu}\left(\lambda\,\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right)=0$$
$$\frac{\partial\beta}{\partial t}+\frac{\partial}{\partial a^\mu}\left(\beta\,\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}\right)=0$$

则有

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\lambda}{\beta}\right)=\frac{\partial}{\partial t}\left(\frac{\lambda}{\beta}\right)+\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\frac{\partial}{\partial a^{\mu}}\left(\frac{\lambda}{\beta}\right)=$$
$$\frac{1}{\beta}\left[\frac{\partial\lambda}{\partial t}+\frac{\partial}{\partial a^{\mu}}\left(\lambda\,\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\right)-\lambda\frac{\partial}{\partial a^{\mu}}\left(\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\right)\right]-$$
$$\frac{\lambda}{\beta^2}\left[\frac{\partial\beta}{\partial t}+\frac{\partial}{\partial a^{\mu}}\left(\beta\,\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\right)-\beta\frac{\partial}{\partial a^{\mu}}\left(\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\right)\right]=$$
$$0$$

于是,命题成立。证毕。

命题5 对于广义经典力学系统(9),如果已知其$(2\omega n-1)$个第一积分,即

$$I_{\sigma}(t,\boldsymbol{a})=C_{\sigma}\quad \sigma=1,\cdots,2\omega n-1 \tag{17}$$

则系统的积分为

$$\int\frac{\lambda^*}{\Delta^*}\left[\mathrm{d}a^{2\omega n}-\left(\Omega^{2\omega n,\nu}\frac{\partial H}{\partial a^{\nu}}\right)^*\mathrm{d}t\right]=C \tag{18}$$

式中:C为常数;Δ为Jacobi行列式

$$\Delta=\frac{\partial(I_1,I_2,\cdots,I_{2\omega n-1})}{\partial(a^1,a^2,\cdots,a^{2\omega n-1})} \tag{19}$$

这里$()^*$表示其中的a^{σ}表示为$a^{2\omega n-1}$和t的函数。

证明 对于广义经典力学系统(9),根据Whittaker给出的一个关系式[14],容易得到

$$\frac{\partial}{\partial a^{\mu}}\left(\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\right)=$$
$$\Delta\left\{\frac{\partial}{\partial a^{2\omega n}}\left[\frac{\left(\Omega^{2\omega n,\nu}\frac{\partial H}{\partial a^{\nu}}\right)^*}{\Delta^*}\right]+\frac{\partial}{\partial t}\left(\frac{1}{\Delta^*}\right)\right\} \tag{20}$$

由于λ是系统的Jacobi最终乘子,有

$$\frac{1}{\lambda}\frac{\bar{\mathrm{d}}\lambda}{\mathrm{d}t}+\frac{\partial}{\partial a^{\mu}}\left(\Omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}}\right)=0 \tag{21}$$

由方程(20,21),得到

$$\frac{1}{\Delta\lambda}\frac{\bar{\mathrm{d}}\lambda}{\mathrm{d}t}+\frac{\partial}{\partial a^{2\omega n}}\left[\frac{\left(\Omega^{2\omega n,\nu}\frac{\partial H}{\partial a^{\nu}}\right)^*}{\Delta^*}\right]+\frac{\partial}{\partial t}\left(\frac{1}{\Delta^*}\right)=0 \tag{22}$$

即有

$$\frac{\partial}{\partial a^{2\omega n}}\left[\frac{\left(\Omega^{2\omega n,\nu}\frac{\partial H}{\partial a^{\nu}}\right)^*\lambda^*}{\Delta^*}\right]+\frac{\partial}{\partial t}\left(\frac{\lambda^*}{\Delta^*}\right)=0 \tag{23}$$

于是表达式

$$\frac{\lambda^*}{\Delta^*}\left[\mathrm{d}a^{2\omega n}-\left(\Omega^{2\omega n,\nu}\frac{\partial H}{\partial a^{\nu}}\right)^*\mathrm{d}t\right]$$

是关于$a^{2\omega n}$和t的某函数的全微分,且式(18)成立。证毕。

3 算 例

设广义经典力学系统的Lagrange函数为[7]

$$L=\frac{1}{2}\alpha_1\dot{q}^2+\frac{1}{2}\alpha_2\ddot{q}^2\quad \alpha_1>0,\ \alpha_2>0 \tag{24}$$

试用Jacobi最终乘子法给出系统的积分。

由式(4,5,7,8),系统的广义Hamilton函数为

$$H=a^2a^3-\frac{1}{2}\alpha_1(a^3)^2+\frac{1}{2\alpha_2}(a^4)^2 \tag{25}$$

运动微分方程(9)给出

$$\dot{a}^1=a^3,\ \dot{a}^2=0,\ \dot{a}^3=\frac{1}{\alpha_2}a^4,\ \dot{a}^4=\alpha_1a^3-a^2 \tag{26}$$

Jacobi最终乘子的确定方程(15)给出

$$\frac{\partial\lambda}{\partial t}+\frac{\partial\lambda}{\partial a^1}a^3+\frac{\partial\lambda}{\partial a^3}\frac{1}{\alpha_2}a^4+\frac{\partial\lambda}{\partial a^4}(\alpha_1a^3-a^2)=0 \tag{27}$$

方程(27)有解

$$\lambda=1 \tag{28}$$
$$\lambda=a^2 \tag{29}$$
$$\lambda=a^2t+a^4-\alpha_1a^1 \tag{30}$$
$$\lambda=a^2a^3-\frac{1}{2}\alpha_1(a^3)^2+\frac{1}{2\alpha_2}(a^4)^2 \tag{31}$$

式(29～31)是所论系统的第一积分。令

$$I_1=a^2=C_1 \tag{32}$$
$$I_2=a^2t+a^4-\alpha_1a^1=C_2 \tag{33}$$
$$I_3=a^2a^3-\frac{1}{2}\alpha_1(a^3)^2+\frac{1}{2\alpha_2}(a^4)^2=C_3 \tag{34}$$

由方程(32～34),得到

$$a^1=\frac{a^4}{\alpha_1}+\frac{C_1}{\alpha_1}t-\frac{C_2}{\alpha_1},a^2=C_1$$
$$a^3=\frac{1}{\alpha_1}\sqrt{\frac{\alpha_1}{\alpha_2}(a^4)^2+(C_1)^2-2\alpha_1C_3}+\frac{C_1}{\alpha_1} \tag{35}$$

计算可得

$$\Delta=\frac{\partial(I_1,I_2,I_3)}{\partial(a^1,a^2,a^3)}=$$
$$\begin{vmatrix}0 & 1 & 0\\ -\alpha_1 & t & 0\\ 0 & a^3 & a^2-\alpha_1a^3\end{vmatrix}=\alpha_1(a^2-\alpha_1a^3) \tag{36}$$

$$\Delta^*=-\alpha_1\sqrt{\frac{\alpha_1}{\alpha_2}(a^4)^2+(C_1)^2-2\alpha_1C_3} \tag{37}$$

式(18)给出积分

$$\int\frac{1}{-\alpha_1\sqrt{\frac{\alpha_1}{\alpha_2}(a^4)^2+(C_1)^2-2\alpha_1C_3}}\cdot$$
$$\left[\mathrm{d}a^4-\sqrt{\frac{\alpha_1}{\alpha_2}(a^4)^2+(C_1)^2-2\alpha_1C_3}\mathrm{d}t\right]=C_4 \tag{38}$$

积分之,可得

$$a^4=\frac{1}{2}\sqrt{\frac{\alpha_1}{\alpha_2}}\mathrm{e}^{-\alpha_1\sqrt{\frac{\alpha_1}{\alpha_2}}C_4}\mathrm{e}^{\sqrt{\frac{\alpha_1}{\alpha_2}}t}-$$

$$\frac{1}{2}\sqrt{\frac{\alpha_1}{\alpha_2}}\left[(C_1)^2-2\alpha_1 C_3\right]\mathrm{e}^{\alpha_1\sqrt{\frac{\alpha_1}{\alpha_2}}C_4}\mathrm{e}^{-\sqrt{\frac{\alpha_1}{\alpha_2}}t} \tag{39}$$

显然,式(35,39)给出了本问题的解。文献[16]用常数变易法也给出了本问题的解,但是显然本文方法更简单。

4　结束语

Jacobi 最终乘子法是求解微分方程的一个十分重要的方法,本文将其推广到广义经典力学系统,为积分广义经典力学系统动力学方程提供了一个新的方法。该方法的主要困难在于:(1) 需要已知系统的足够数量的相互独立的第一积分;(2) 求确定方程(15)的解。但是,只要已知系统的($2\omega n-1$)个第一积分,并找到其 Jacobi 最终乘子,便可立即给出系统的运动。算例表明本文方法的有效性和满足上述条件的情况下该方法的优越性。

参考文献:

[1] Podolsky B. A generalized electrodynamics[J]. Phys Rev, 1942, 65: 228-235.

[2] De León M, Rodrigues P R. Generalized classical mechanics and field theory[M]. Amsterdam, North-Holland: Elservier Science Publishers B V, 1985.

[3] 李子平. 经典和量子约束系统及其对称性质[M]. 北京: 北京工业大学出版社, 1993.
Li Ziping. Classical and ouantal dynamics of constrained systems and their symmetrical properties [M]. Beijing: Beijing Polytechnic University Press, 1993.

[4] 梅凤翔. 李群和李代数对约束力学系统的应用[M]. 北京: 科学出版社, 1999.
Mei Fengxiang. Applications of Lie groups and Lie algebras to constrained mechanical systems [M]. Beijing: Science Press, 1999.

[5] Li Z P, Jiang J H. Symmetries in constrained canonical systems[M]. Beijing: Science Press, 2002.

[6] Qiao Y F, Yue Q W, Dong Y A. Noether's conservation laws of holonomic nonconservative dynamical systems in generalized mechanics [J]. Appl Math Mech, 1994, 15(9): 877-883.

[7] Zhang Y, Shang M, Mei F X. Symmetries and conserved quantities for systems of generalized classical mechanics[J]. Chin Phys, 2000, 9(6): 401-407.

[8] 乔永芬, 李仁杰, 赵淑红. 高维增广相空间中广义力学系统的对称性和不变量[J]. 物理学报, 2001, 50(5): 811-815.
Qiao Yongfen, Li Renjie, Zhao Shuhong. Symmetry and invariant in generalized mechanical systems in the high-dimensional extended phase space[J]. Acta Phys Sin, 2001, 50(5): 811-815.

[9] Qiao Y F, Zhang Y L, Han G C. Integrating factors and conservation theorem for holonomic nonconservative dynamical systems in generalized classical mechanics[J]. Chin Phys, 2002, 11(10): 988-992.

[10] 张毅, 梅凤翔. 广义经典力学系统对称性的摄动与绝热不变量[J]. 物理学报, 2003, 52(10): 2368-2372.
Zhang Yi, Mei Fengxiang. Perturbation to symmetries and adiabatic invariant for systems of generalized classical mechanics[J]. Acta Phys Sin, 2003, 52(10): 2368-2372.

[11] Qiao Y F, Zhao S H. Integrating factors and conservation theorems of Lagrangian equations for nonconservative mechanical system in generalized classical mechanics[J]. Commun Theor Phys, 2006, 46(1): 43-45.

[12] Zhang Y. A new type of adiabatic invariants for nonconservative systems of generalized classical mechanics [J]. Chin Phys, 2006, 15(9): 1935-1940.

[13] Zhang P Y, Fang J H, Wang P, et al. Non-Noether conserved quantity of Poincaré-Chetaev equations of a generalized classical mechanics[J]. Commun Theor Phys, 2006, 45(6):961-964.

[14] Whittaker E T. A treatise on the analytical dynamics of particles and rigid bodies[M]. Fourth Edition. Cambridge: Cambridge University Press, 1952.

[15] Mei F X, Shang M. Last multiplier of generalized Hamilton system [J]. Chin Phys Lett, 2008, 25(11): 3837-3839.

[16] 张耀良,韩广才. 求解广义经典力学方程的常数变易法[J]. 哈尔滨工程大学学报,1999,20(1):78-84.
Zhang Yaoliang, Han Guangcai. The method of variation of parameters for solving the equations of generalized classical mechanics[J]. Journal of Harbin Engineering University, 1999,20(1):78-84.